

# *Mathematical Journal of Okayama University*

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*Volume 3, Issue 2*

1953

*Article 7*

MARCH 1954

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## On the regularly convex hull of a set in a conjugate Banach space

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# ON THE REGULARLY CONVEX HULL OF A SET IN A CONJUGATE BANACH SPACE

MINORU TOMITA

The convex hull of a finite set  $(p_1, p_2, \dots, p_n)$  in a linear space is the set of all  $\sum_{i=1}^n \alpha_i p_i$  so that  $\alpha_i \geq 0$  and  $\sum \alpha_i = 1$ . In this paper we shall observe an analogous representation for the regularly convex hull of a bounded weakly closed set in the conjugate space of a Banach space.

**Theorem 1.** *Let  $\mathbf{E}$  be a real Banach space, and  $\mathfrak{X}$  be a bounded weakly closed sub-set of the conjugate Banach space. Let  $\text{Co}(\mathfrak{X})$  denote the smallest regularly convex set which contains  $\mathfrak{X}$ . Then  $\text{Co}(\mathfrak{X})$  is the set of all  $\int_{\mathfrak{X}} \lambda d\varphi(\lambda)$  so that  $\varphi$  are non-negative Borel measures on  $\mathfrak{X}$  with  $\varphi(\mathfrak{X}) = 1$ , where  $p = \int_{\mathfrak{X}} \lambda d\varphi(\lambda)$  are Pettis integrals defined by  $p(A) = \int_{\mathfrak{X}} \lambda(A) d\varphi(\lambda)$ .*

*Proof.* Put  $A^\nu(\lambda) = \lambda(A)$  for every  $A \in \mathbf{E}$  and  $\lambda \in \mathfrak{X}$ .  $\mathfrak{X}$  is bounded and contained in a sphere ( $|\varphi| \leq r$ ). Then we have  $|A^\nu(\lambda)| = |\lambda(A)| \leq r|A|$ . And  $A \rightarrow A^\nu$  is a bounded linear transform of  $\mathbf{E}$  in the Banach space  $\mathbf{R}$  of all real weakly continuous functions on  $\mathfrak{X}$ . For every linear functional  $f$  on  $\mathbf{R}$  define the linear functional  $f^\nu$  on  $\mathbf{E}$  by  $f^\nu(A) = f(A^\nu)$ . Then  $f \rightarrow f^\nu$  is the conjugate transform of  $A \rightarrow A^\nu$ , and it is weakly continuous. Let  $\mathfrak{P}$  denote the set of all linear functionals  $p$  on  $\mathbf{R}$  so that  $p(I) = 1$  and  $p(f) \geq 0$  for every  $0 \leq f \in \mathbf{R}$ . Let  $\mathfrak{N}$  denote the set  $(\delta_\lambda: \lambda \in \mathfrak{X})$ , where  $\delta_\lambda$  denotes the linear functional on  $\mathbf{R}$  so that  $\delta_\lambda(f) = f(\lambda)$  for each fixed point  $\lambda \in \mathfrak{X}$ . Then clearly  $\mathfrak{P} \supseteq \mathfrak{N}$ . We show that  $\text{Co}(\mathfrak{N}) = \mathfrak{P}$ . Let  $f \in \mathbf{R}$ , then every  $p \in \mathfrak{P}$  satisfies  $|p(f)| \leq |f|$ . But from the weak compactness of the set  $\mathfrak{X}$ , there exists  $\lambda \in \mathfrak{X}$  so that  $|\delta_\lambda(f)| = |f(\lambda)| = \sup_{\mu \in \mathfrak{X}} |f(\mu)| = |f|$ .

Hence by the theorem of Krein-Smulian,  $\text{Co}(\mathfrak{N})$  contains  $\mathfrak{P}$ . Since  $\mathfrak{P}$  is weakly closed and convex, it is regularly convex. Then  $\text{Co}(\mathfrak{N}) = \mathfrak{P}$ . Now  $p \rightarrow p^\nu$  maps  $\mathfrak{P}$  on the smallest weakly compact convex set which contains all  $(\delta_\lambda^\nu: \delta_\lambda \in \mathfrak{N})$ . But every  $\delta_\lambda^\nu$  at  $\lambda \in \mathfrak{X}$  coincides with  $\lambda$ . Then the image  $(p^\nu: p \in \mathfrak{P})$  of  $\mathfrak{P}$  coincides with  $\text{Co}(\mathfrak{X})$ .

Now it is well-known that every positive functional  $p$  on  $\mathbf{R}$  is an

indefinite integral  $p(f) = \int f d\varphi$ , where  $\varphi$  is a non-negative Borel measure on  $X$ . Then every  $p^\nu$  for  $p \in \mathfrak{P}$  satisfies

$$p^\nu(A) = p(A^\nu) = \int_{\mathfrak{K}} A^\nu(\lambda) d\varphi(\lambda) = \int_{\mathfrak{K}} \lambda(A) d\varphi(\lambda).$$

Hence  $p^\nu = \int \lambda d\varphi(\lambda)$ , where  $\varphi(\mathfrak{K}) = p(1) = 1$ . This concludes the theorem.

Let  $\mathfrak{K}$  be a bounded regularly convex set in the conjugate space of a Banach space. By the theorem of Krein-Milman,  $\mathfrak{K}$  is the smallest regularly convex set which contains all the extreme points of  $\mathfrak{K}$ . Then we obtain

**Theorem 2.** *Let  $\mathfrak{K}$  be a bounded regularly convex set in the conjugate space of a Banach space. Let  $\mathfrak{E}$  denote the set of all extreme points of  $\mathfrak{K}$ , and  $\overline{\mathfrak{E}}$  denote its weak closure. Then every  $f \in \mathfrak{K}$  is expressible by a Pettis integral*

$$f = \int_{\overline{\mathfrak{E}}} \lambda d\varphi(\lambda),$$

where  $\varphi$  is a non-negative Borel measure on  $\overline{\mathfrak{E}}$  with  $\varphi(\overline{\mathfrak{E}}) = 1$ .

Let  $\mathbf{A}$  be a uniformly closed self-adjoint algebra of operators on a Hilbert space which contains the identity  $I$ . A linear functional  $f$  on  $\mathbf{A}$  is said a state if it satisfies  $f(A^*) = \overline{f(A)}$  and  $f(A^*A) \geq 0$ . Then the set  $\mathfrak{P}$  of all states  $p$  with  $p(I) = 1$  is a bounded regularly convex set conjugate to the real Banach space  $\mathbf{A}^n$  of all Hermitian operators in  $\mathbf{A}^\nu$ . A state  $p$  is said irreducible if there is no pair  $(q, r)$  of states with  $p = q + r$  other than  $q = \alpha p$  and  $r = (1 - \alpha)p$ . Now  $p \in \mathfrak{P}$  is irreducible if and only if it is an extreme point of  $\mathfrak{P}$ . Therefore

**Theorem 3.** *Let  $\mathbf{A}$  be a uniformly closed self-adjoint algebra of operators on a Hilbert space which contains the identity  $I$ . Let  $\mathfrak{N}$  denote the set of all irreducible states  $u$  with  $u(I) = 1$ , and  $\overline{\mathfrak{N}}$  denote its weak closure. Then every state  $p$  on  $\mathbf{A}$  is expressed by a Pettis integral*

$$p = \int_{\overline{\mathfrak{N}}} \lambda d\varphi(\lambda),$$

where  $\varphi$  is a suitable non-negative Borel measure on  $\overline{\mathfrak{N}}$ .

In fact, every state  $p$  with  $p(I) = 1$  is expressed as the Theorem using Theorem 2. But every state  $p$  is denoted as  $\alpha q$ , where  $q$  is a state with  $q(I) = 1$ . Then Theorem 3 is valid.

The last theorem is closely related to my immediately subsequent paper of this Journal.

**Footnote.**

1). Let  $p$  be a state. Given an Hermitian  $0 \neq A \in \mathbf{A}$ , we put  $B = |A|^{\frac{1}{2}}$  ( $\sum_0^{\infty} c_t (A/|A|)^t$ ), where  $\sum_0^{\infty} c_t x^t$  is the power-series expansion of  $(1-x)^{\frac{1}{2}}$ .  $B$  converges uniformly, and satisfies  $B^*B = |A|I - A$ . Then  $p(A) \leq |A|p(I)$ . Hence  $p$  is a bounded linear functional on  $\mathbf{A}^H$  with  $|p| = p(I)$ . Therefore  $\mathfrak{P}$  is a bounded weakly closed convex subset of the conjugate space of  $\mathbf{A}^H$ .

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*(Received December 20, 1953)*